Missing Responses in Longitudinal Data

Example: Study on the Financial Crisis

The data that we analyzed...

id	Age	Sex	Y2	Y3	Y4	Y5	Y6	Y7	Y8
10	23.92	F	2	2	3	2	3	3	3
17	76.09	M	2	4	2	4	4	4	4
29	54.04	F	1	2	2	4	3	3	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1810	46.77	M	2	2	2	2	2	2	2
1826	24.77	M	2	2	2	2	2	3	1
1837	26.29	F	1	1	1	1	1	1	1

Example: Study on the Financial Crisis

The data that were collected...

id	Age	Sex	Y2	Y3	Y4	Y5	Y6	Y7	Y8
2	40.55	F	2	1	3	2	NA	3	NA
10	23.92	F	2	2	3	2	3	3	3
11	44.15	M	NA	NA	NA	NA	1	3	NA
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2439	29.58	M	4	4	NA	NA	2	4	NA
2441	54.64	M	2	1	1	1	1	1	NA
2612	30	M	2	1	1	NA	NA	NA	NA

Missing data refer to any observations which we *intended* to collect, but which were not recorded in our data file for **any reason**.

Missing data is a pervasive problem across most domains, but it is *particularly* common in longitudinal studies.

Classification of Missing Data Mechanisms

Notation for Missingness

We define an **observation indicator** R_{ij}.

$$R_{ij} = \begin{cases} 1 & Y_{ij} \text{ is observed,} \\ 0 & \text{otherwise.} \end{cases}$$

We will assume that we are only concerned with missingness in the **outcomes**, and that R_{ij} is observed for all i, j.

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We partition the **outcome** Y_i into the **observed components** Y_i^{O} and the **missing components**, Y_i^{M}.

A Hierarchy of Types of Missing Data

- Data are said to be **missing completely at random (MCAR)** if

$$f_{R_i}(r_i|Y_i^{\mathrm{O}}, Y_i^{\mathrm{M}}, X_i) = f_{R_i}(r_i|X_i).$$

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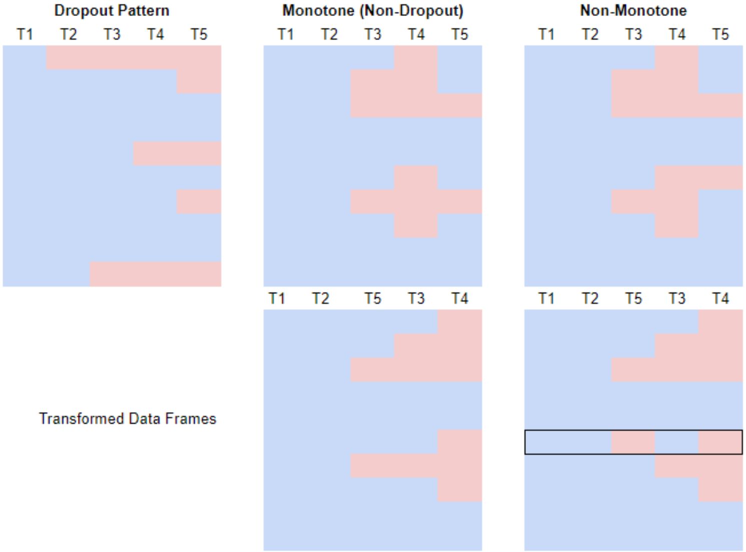
- Otherwise, data are said to be **not missing at random (NMAR)**.
  - Data may be NMAR if, for instance, individuals who smoke more (Y_{ij} large) are less likely to continue responding to a smoking questionnaire.

Patterns of Missingness



Impacts of Missingness

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4. In all other situations, estimators will be **biased**, and inference will be **invalid**.

General Techniques for Handling Missingness

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3. **Weighting techniques**, where pseudo datasets are created based on weighting the available information.
4. **Imputation techniques**, where the missing values are filled-in based on an underlying model.

A Note on Handling NMAR Missingness

The four classes of techniques listed above will **not** be valid for NMAR data. NMAR data **need** joint modelling strategies for

$$f(Y_i, R_i).$$

Weighting Techniques

Dropout as Missingness

One special type of missingness is **dropout**. In this case, an individual is observed only until t_j, and no times after.

Define D_i to be the **dropout time**

$$D_i = 1 + \sum_{j=1}^{K} R_{ij}.$$

Probability of Inclusion

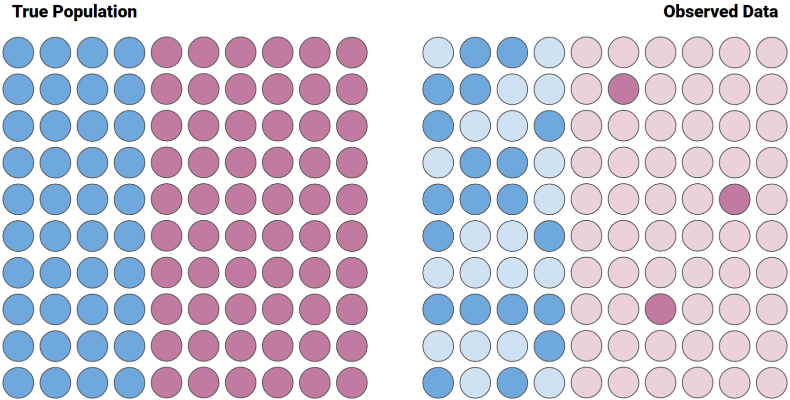
We can think about estimating the **probability of inclusion** for any individual, at any time in the study.

$$\pi_{ij} = P(D_i > j | D_i \geq j, i).$$

This gives the probability that individual i was still under observation at time j, assuming that they made it to at least time $j - 1$. We can estimate these probabilities via (e.g.) logistic regression.

Individuals with **low** π_{ij} were unlikely to have been observed. Individuals with **high** π_{ij} were likely to have been observed.

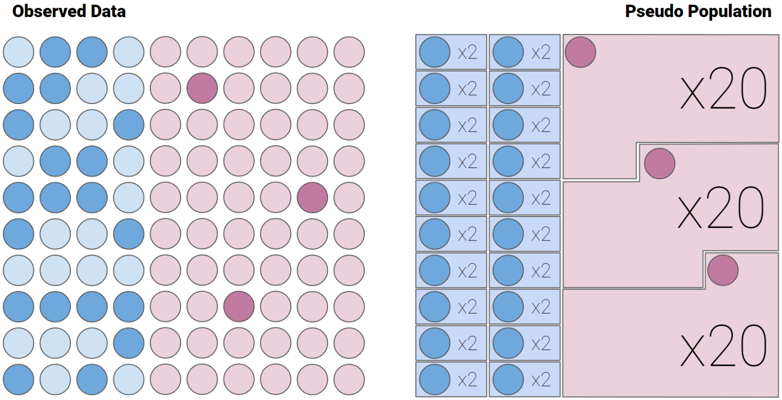
Balancing the Observed Data



$$\pi_B = P(\text{Observed}|\text{Blue}) = \frac{20}{40} = 0.5 \quad \text{and} \quad \pi_M = P(\text{Observed}|\text{Magenta}) = \frac{3}{60} = 0.05.$$

Constructing a Pseudo-Population

$$w_B = \frac{1}{\pi_B} = 2 \quad \text{and} \quad w_M = \frac{1}{\pi_M} = 20.$$



Applying this to Longitudinal Data

We can take

$$\pi_i = P(D_i > K) = \prod_{j=1}^{K} \pi_{ij},$$

and correspondingly get

$$w_i = \frac{1}{\pi_i} = \frac{1}{\prod_{j=1}^{K} \pi_{ij}}.$$

Then, by running a **complete case analysis** with these weights, any MCAR or MAR missingness will be accounted for (assuming the π_i are correct).

More Efficiently

We can extend this idea to weighting **all available data** instead of just the complete cases.

$$w_{ij} = \frac{1}{P(D_i > j)} = \left[\prod_{\ell=1}^{j} \pi_{i\ell} \right]^{-1}.$$

Then giving each individual's observations at time j a weight of w_{ij}, and running an **available data analysis** produces more efficient corrected estimators.

IPW-GEE

Applied specifically to GEE:

- Define $W_i = \text{diag}(R_{ij} w_{ij} \mid j = 1, \ldots, K_i)$.
- Find $\widehat{\beta}$ which solve

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  - Recall that V_i need not be correctly specified. If X_i is not fully observed, take V_i diagonal.

Imputation Techniques

Imputation in General

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Need to choose **single** or **multiple** imputation, and the **imputation procedure**.

Multiple Imputation

Repeat the imputation process m times, giving $\widehat{\beta}^{(k)}$ for $k = 1, \ldots, m$. Then

$$\widehat{\beta} = \frac{1}{m} \sum_{k=1}^{m} \widehat{\beta}^{(k)},$$

and we further take

$$\widehat{\text{cov}}\left(\widehat{\beta}\right) = \frac{1}{m} \sum_{k=1}^{m} \text{cov}\left(\widehat{\beta}^{(k)}\right) + \frac{m+1}{m(m-1)} \sum_{k=1}^{m} \left(\widehat{\beta}^{(k)} - \widehat{\beta}\right)\left(\widehat{\beta}^{(k)} - \widehat{\beta}\right)'.$$

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5. Continue imputing, estimate $\widehat{\beta}^{(1)}$, and then repeat m times.

Problem with Underestimating Uncertainty

This procedure outlined **underestimates** the variability that should be inherent to this imputation procedure, since $\widehat{Y}_{ij}$ is estimated not fixed!

Instead of using estimated $\widehat{\gamma}_j$, we **draw from the posterior distribution**, giving $\widetilde{\gamma}_j$, and otherwise proceed as outlined.

Predictive Mean Matching

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4. Repeat this process for all j, and then m times.

Likelihood as Imputation

Any technique that uses **maximum likelihood** (e.g., GLMEMs or transition models) will result in valid inference if the data are **MCAR** or **MAR**. In this case

$$f(Y_i|X_i) = f(Y_i^{\mathrm{O}}|X_i) = f(Y_i^{\mathrm{M}}|X_i).$$

There are procedures (using Expectation Maximization (EM)) which make the connection between **likelihood** and **imputation** more clear.

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- **Weighting techniques** generate pseudo-populations that match the would-be observed population using estimated probabilities.
- **Imputation techniques** fill in the missing values based on specific regression models.